6.3 Ecological Models (continued)

last time: predator-prey $\quad x' = x(a - py)$ $\quad$ y uses x as food

$$y' = y(-b + gx)$$

competition system: two species go after a common food but not each other (e.g. squirrels and chipmunks)

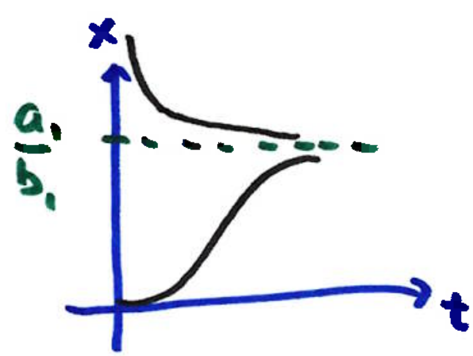
$$x' = a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_1 x - c_1 y)$$

$$y' = a_2 y - b_2 y^2 - c_2 xy = y(a_2 - b_2 y - c_2 x)$$

$$a_i,\ b_i,\ c_i > 0$$

in the absence of y, $x' = x(a_1 - b_1 x)$

logistic growth: grow until the carrying capacity then stabilizes



introduction of y slows down growth and lowers carrying capacity (same for y) example : $x' = x(1-x-y)$

$y' = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$

cp: $(0,0)$, $(1,0)$, $(0,\frac{3}{4})$, $(\frac{1}{2},\frac{1}{2})$

both die | y dies | x dies | coexistence

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -\frac{1}{2}y & \frac{3}{4}-2y-\frac{1}{2}x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \quad \lambda = 1, \frac{3}{4} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

source, unstable

$$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix} \quad \lambda = -1, \frac{1}{4} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

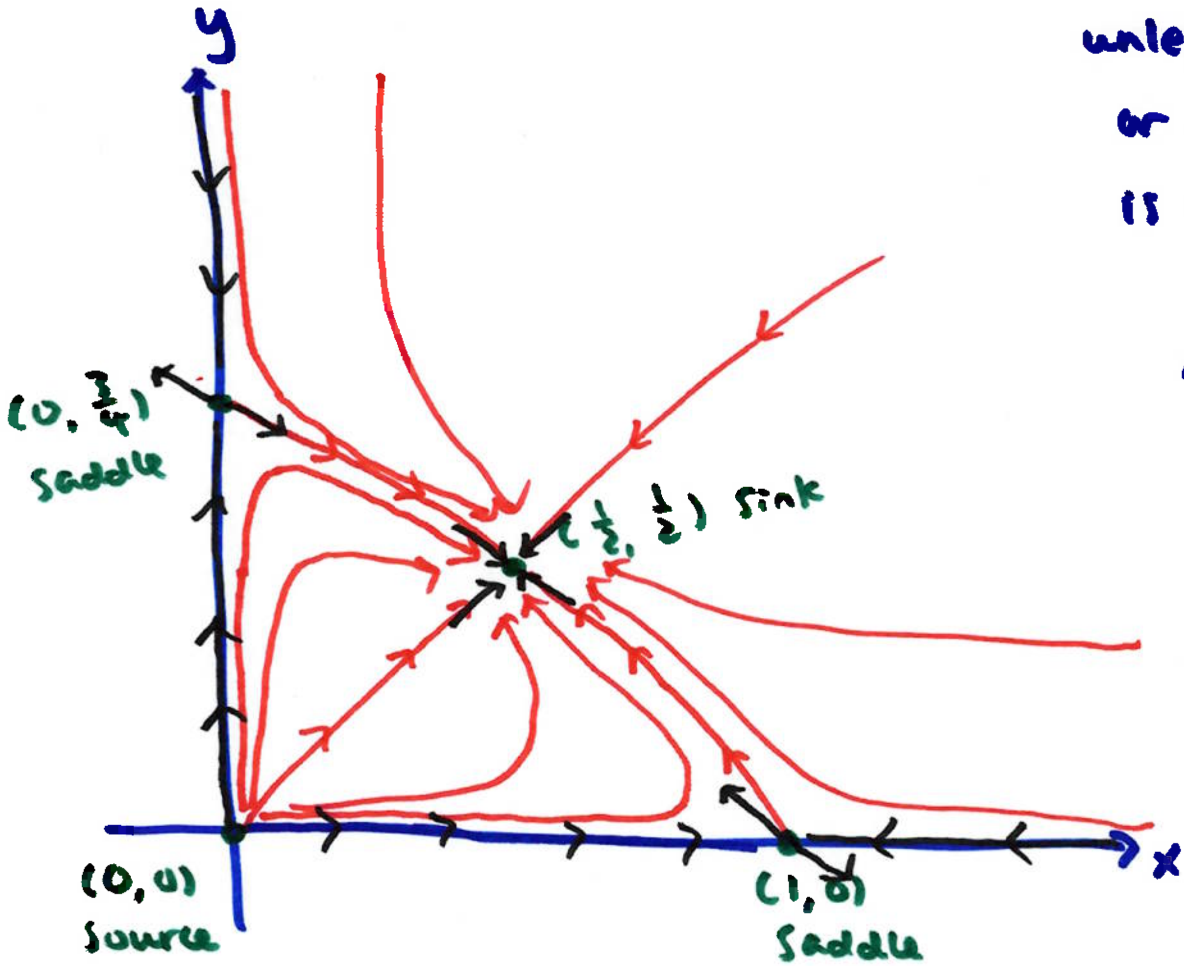
saddle, unstable

$$J(0,\tfrac{3}{4}) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix} \quad \lambda = \frac{1}{4}, -\frac{3}{4} \quad \vec{v} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

saddle

$$J(\tfrac{1}{2},\tfrac{1}{2}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \quad \lambda = -0.15, -0.85 \quad \vec{v} = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

sink, asymp. stable



unless either x(0), y(0) or both are 0, coexistence is the eventual outcome

↓

"weak" competition

both settle down at level below their intrinsic carrying capacities

example

$$x' = x(1-x-y)$$

$$y' = y\left(\tfrac{1}{2} - \tfrac{1}{4}y - \tfrac{3}{4}x\right)$$

cp: $(0,0)$, $(1,0)$, $(0,2)$, $\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -\tfrac{3}{4}y & \tfrac{1}{2} - \tfrac{1}{2}y - \tfrac{3}{4}x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \tfrac{1}{2} \end{bmatrix} \qquad \lambda = 1, \tfrac{1}{2}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{source}$$

$$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & -\tfrac{1}{4} \end{bmatrix} \qquad \lambda = -1, -\tfrac{1}{4}$$

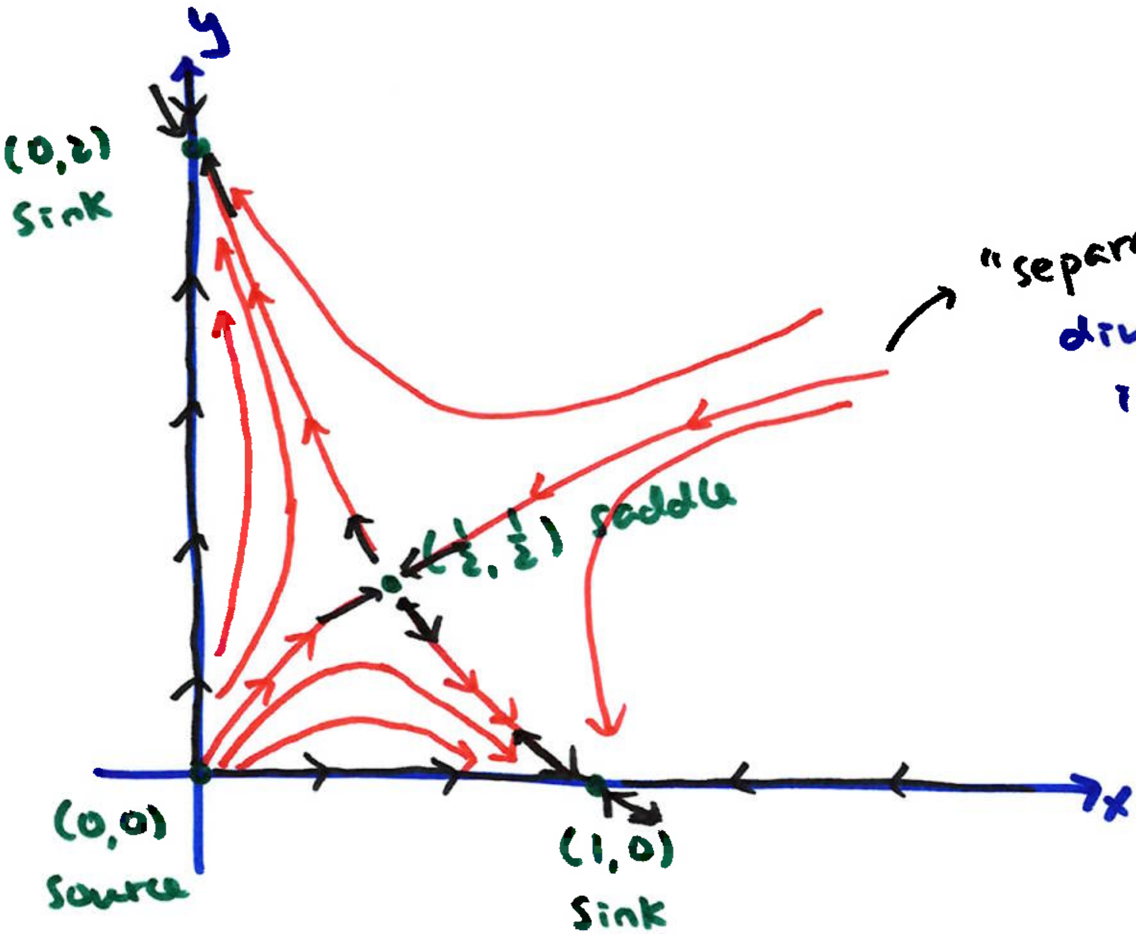
$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad \text{sink}$$

$$J(0,2) = \begin{bmatrix} -1 & 0 \\ -\tfrac{3}{2} & -\tfrac{1}{2} \end{bmatrix} \qquad \lambda = -1, -\tfrac{1}{2}$$

$$\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{sink}$$

$$J\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{bmatrix} -\tfrac{1}{2} & -\tfrac{1}{2} \\ -\tfrac{3}{8} & -\tfrac{1}{8} \end{bmatrix} \qquad \lambda = 0.16, -0.78$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \qquad \text{saddle}$$



"separatrix"

divides the regions where initial conditions lead to different outcomes

eventual outcome is the extinction of one

"strong" competition

$$x' = a_1 x - b_1 x^2 - c_1 x y$$

$$y' = a_2 y - b_2 y^2 - c_2 x y$$

$$a_i, b_i, c_i > 0$$

coexistence is likely ("weak" competition) if

$$b_1 b_2 > c_1 c_2$$

intrinsic limits are more important than interactions if $c_i < 0$ → cooperation system

(ants and aphids)

interaction boosts the carrying capacities

coexistence is most likely

but may lead to "doomsday" scenario

→ when both x and y go to ∞